Group actions and Cayley Graphs

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§0 Motivation:

- Groups occur in many different areas of mathematics.

Examples: → $\mathrm{Homeo}(X)$
→ $\mathrm{Sym}(X)$
→ Lie Groups
→ "C^*-Algebras"
→ $\pi_k(X)$
→ ... many more!

Problem: It is very hard to understand groups via their combinatorics!

Idea: Embed G into the symmetries of a space with rich geometry

→ **Geometric Group Theory!**

5 Minuten ca

§1 Reminder: Group Theory

Convention: Unless stated otherwise, we write groups multiplicatively with neutral element $\mathbb{1}$.

Q: Can we find an "analogue" of a basis of a vector space for groups?

Def: Let G be a group. A subset $S \subseteq G$ is called generating set for G if: For every $1 \neq g \in G$ there exist $s_1, \dots, s_n \in S$, $e_1, \dots, e_n \in \mathbb{Z} \setminus \{0\}$, $s_i \neq s_{i+1}$: $g \overset{(*)}{=} s_1^{e_1} s_2^{e_2} \cdots s_n^{e_n}$. We write $G = \langle S \rangle$

Examples:

- $G = \langle G \rangle$
- $\mathbb{Z} = \langle 1 \rangle$
- $\mathbb{Z} = \langle 2, 3 \rangle$

Reminder: If for a group G there exists a generating set S such that $(*)$ is unique for every $1 \neq g \in G$, then we call G a free group, write $\mathcal{F}(S)$ for this gp.

Def: $\langle S \mid R \rangle := \mathcal{F}(S) / \langle\langle R \rangle\rangle$, $\quad R \subseteq \mathcal{F}(S)$, $\mathcal{F}(S)$ free gp. over S, $\quad \langle\langle R \rangle\rangle := \bigcap \{ N \trianglelefteq \mathcal{F}(S) \mid R \subseteq N \}$

this is called a group presentation.

Example:

- $\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1} \rangle$, $\quad aba^{-1}b^{-1} = 1 \iff ab = ba$.
- $\mathcal{F}(X) = \langle X \mid \emptyset \rangle$
- $\mathrm{Sym}(3) = \langle a, b \mid a^2, b^2, (ab)^3 \rangle$.

15 min.

- $F(X) = \langle X \mid \emptyset \rangle$
- $Sym(3) = \langle a, b \mid a^2, b^2, (ab)^3 \rangle$. 15 min.

§2 Group Actions

Def: Let G be a gp. and X a set.

A group action of G on X is a map $\varphi: G \times X \to X$ such that:

① $\varphi(1, x) = x \quad \forall x \in X$ (1 → neutral element!)

② $\varphi(g \cdot h, x) = \varphi(g, \varphi(h, x)) \quad \forall g, h \in G, x \in X$.

Ex: ① $Sym(n) \curvearrowright \{x_1, \ldots, x_n\}$ via permutation

② If X is a topological space, then $Homeo(X) \curvearrowright X$ via
$(g, x) \mapsto g(x)$

③ $G \curvearrowright G$ via

ⓐ $(g, h) \mapsto g \cdot h$

ⓑ $(g, h) \mapsto g h g^{-1}$.

④ $\mathbb{Z}^n \curvearrowright \mathbb{R}^n$
$(z, r) \mapsto z + r$.

Some vocabulary:

Let $\varphi: G \times X \to X$ be a group action.

① The orbit $G(x)$ of a point $x \in X$ is $G(x) := \{ \varphi(g, x) \mid g \in G \}$.

② The stabilizer $Stab_G(x)$ of a point $x \in X$ is $Stab_G(x) = \{ g \in G \mid \varphi(g, x) = x \}$.

③ $F \subseteq X$ is a fundamental domain if $\forall x \in X \ \#(G(x) \cap int(F)) = 1$.

We say φ is...

Ⓐ transitive if $\exists x \in X: G(x) = X$.

Ⓑ cocompact if $\exists C \subseteq X$ cpt.: $G(C) = X$.

Ⓒ proper (→ properly discontinuous) if X is a metric space and $\forall x \in X \ \exists \varepsilon > 0$:

$$\# \{ g \in G \mid \varphi(g, B_\varepsilon(x)) \cap B_\varepsilon(x) \neq \emptyset \} < \infty$$

Ⓓ free if $\forall x \in X, g \in G \setminus \{1\}$: $\varphi(g, x) \neq x$.

(D) free if $\forall x \in X,\ g \in G \setminus \{1\}$:
$\varphi(g, x) \neq x$.

(E) faithful if $\forall g \neq h \in G\ \exists\ x \in X$:
$\varphi(g,x) \neq \varphi(h,x)$.

(F) Simply transitive if φ is free + transitive.

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Ex: ① from before is faithful but not free

④ from before is proper + cocompact. [we call that "geometric"].

⋮

Q: Why is that useful?

A: E.g.:

Thm: $G \curvearrowright$ Tree freely [+without inversion]
$\Leftrightarrow$ G free

Thm: $G \curvearrowright \mathbb{R}^n$ proper + cocompact
$\Rightarrow$ G virt. abelian.

$\rightsquigarrow$ many more Thms exist!

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§3 Cayley Graph

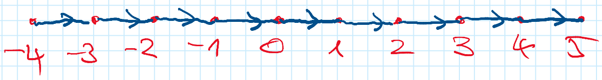
Q: Starting with an arbitrary group, how do we come up with these spaces?

Def: Cayley Graph: let $G = \langle S \rangle$ be a group, assume $1 \notin S$. Define a graph $\Gamma = (V, E)$ by:

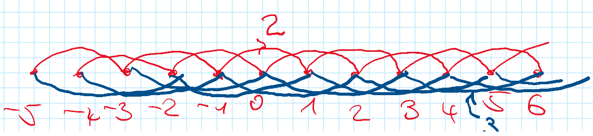
$$V = G$$
$$E = \{ (x, x \cdot s) \mid x \in V,\ s \in S \}$$

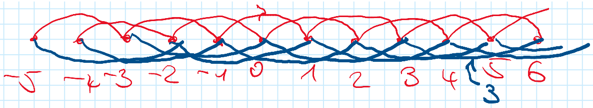
if $\operatorname{ord}(s) \neq 2$, then $(x, x \cdot s)$ has a direction.
This is the Cayley-Graph $\operatorname{Cay}(G, S)$.

Ex: ① $\mathbb{Z} = \langle 1 \rangle$

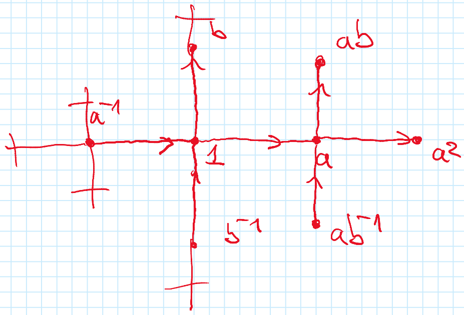


② $\mathbb{Z} = \langle 2, 3 \rangle$

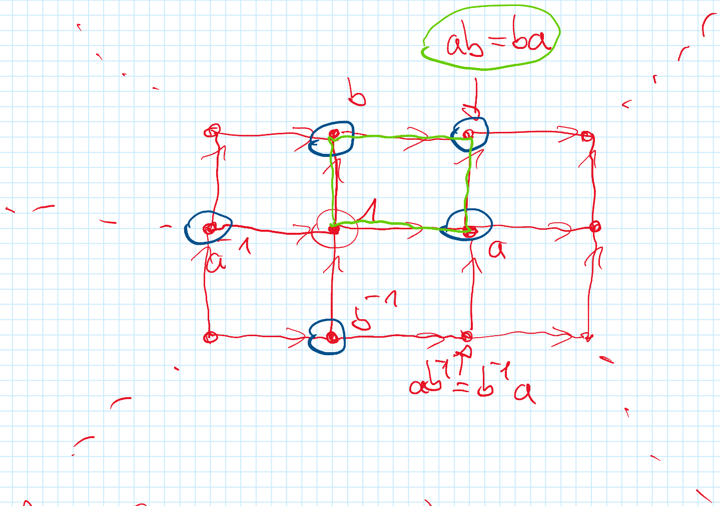




③ $F(a,b) = \langle a, b \rangle$.
$\rightsquigarrow$ 4-valent tree



④ $\mathbb{Z}^2 = \langle a, b \rangle$



Remark: $G \curvearrowright Cay(G,S)$ via
$\varphi(g,v) := g\cdot v \qquad v\in V$
$\varphi(g,(x,x\cdot s)) = (g\cdot x, g\cdot x\cdot s) \qquad (x,x\cdot s)\in E.$

Thm: φ is sharply transitive on the vertices!

Rmk: Given $\varphi: G \curvearrowright \Gamma$, Γ connected graph, φ sharply transitive on the vertices, we can obtain a nice generating set S for the group by "reversing" the construction. Then $\Gamma \cong Cay(G,S)$.

Def: The word metric on G:
Let $G = \langle S \rangle$.
Let $w \in G$. Define
$$\ell_S(w) = \min\{n\in\mathbb{N} \mid g = s_1^{\varepsilon_1}\cdots s_n^{\varepsilon_n}, \boxed{\varepsilon_i\in\{\pm1\},}\ \underset{s_i\in S}{} \}$$

$$\ell_S(w) = \min\{n \in \mathbb{N} \mid g = s_1^{\varepsilon_1} \cdots s_n^{\varepsilon_n}, \boxed{\varepsilon_i \in \{\pm 1\}}, s_i \in S\}$$

and $d(g,h) := \ell_S(g^{-1}h) \rightsquigarrow$ word-metric

Rmk: This is precisely the metric we obtain by assigning length 1 to each edge in the Cayley Graph!

§4 Presentation complex

Idea: Extend the definition of a Cayley-Graph to a 2-complex using the relations.
$\rightsquigarrow$ This is a bit more involved.

Thm: $G = \langle S | R \rangle \Rightarrow \exists$ simply connected 2-complex Δ, $G \curvearrowright \Delta$, the action is simply transitive on the vertices, 1-skel. is $Cay(G,S)$.
Conversely given such a 2 complex and gp action, we can "read off" a presentation for the gp.

Recap: • Given a group $G = \langle S \rangle$ or $G = \langle S | R \rangle$, there are 2 nice natural spaces, on which G acts nicely, $Cay(G,S)$, and the presentation complex.
• Gp actions can be good tools to understand groups.

Goal: Given $G = \langle S | R \rangle$, such that each $s_i \in S$ has order 2, we want to extend the definition of the presentation complex to obtain an even nicer space with an even better group action!